The sharp constant in the Hardy-Sobolev-Maz'ya inequality in the three dimensional upper half-space

Rafael D. Benguria[1], Rupert L. Frank[2] and Michael Loss[3]

1. Department of Physics, P. Universidad Católica de Chile, Casilla 306, Santiago 22, Chile, email: rbenguri@fis.puc.cl

2. Matematiska institutionen, KTH Stockholm, 100 44 Stockholm, Sweden, email: rupert@math.kth.se

3. School of Mathematics, Georgia Tech, Atlanta, GA 30332 email: loss@math.gatech.edu

May 7, 2007

Abstract

It is shown that the sharp constant in the Hardy-Sobolev-Maz'ya inequality on the upper half space $\mathbb{H}^3 \subset \mathbb{R}^3$ is given by the Sobolev constant. This is achieved by a duality argument relating the problem to a Hardy-Littlewood-Sobolev type inequality whose sharp constant is determined as well.

1 Introduction

The present work is concerned with a particular case of the Hardy-Sobolev-Maz'ya inequality

$$\int_{\mathbb{H}^n} \left[|\nabla f(\mathbf{x})|^2 - \frac{1}{4y^2}|f(\mathbf{x})|^2 \right] d\mathbf{x} \geq C_n \left(\int_{\mathbb{H}^n} |f(\mathbf{x})|^{\frac{2n}{n-2}} d\mathbf{x} \right)^{\frac{n-2}{n}} \tag{1}$$

[1]Work partially supported by Fondecyt (CHILE) projects 106–0651 and 706–0200, and CONICYT/PBCT Proyecto Anillo de Investigación en Ciencia y Tecnología ACT30/2006.

[2]Work partially supported by the Swedish Foundation for International Cooperation in Research and Higher Education (STINT).

[3]Work partially supported by NSF-grant DMS-0600037.

where f is a compactly supported function that lives in the half space

$$\mathbb{H}^n := \{\mathbf{x} = (x, y) : x \in \mathbb{R}^{n-1}, y > 0\} \ . \tag{2}$$

It is quite easy to see that the left side is positive; this is Hardy's inequality. That (1) holds for a strictly positive constant C_n was proved by Maz'ya [9] (Section 2.1.6., Corollary 3). In what follows, C_n denotes the sharp constant in the above inequality. It was shown in recent work by Tertikas and Tintarev [10], that an optimizer for the sharp constant C_n exists provided the dimension $n \geq 4$.

The functional (1) has a number of equivalent formulations. For once it is equivalent to the inequality

$$\int_{\mathbb{B}^n} |\nabla g(\mathbf{\Omega})|^2 d\mathbf{\Omega} - \int_{\mathbb{B}^n} \frac{1}{(1-|\mathbf{\Omega}|^2)^2} |g(\mathbf{\Omega})|^2 d\mathbf{\Omega} \geq C_n \left(\int_{\mathbb{B}^n} |g(\mathbf{\Omega})|^{\frac{2n}{n-2}} d\mathbf{\Omega} \right)^{\frac{n-2}{n}} \tag{3}$$

where $\mathbb{B}^n$ is the unit ball in $\mathbb{R}^n$. To see this, set

$$f(x, y) = \left(\frac{2}{(1+y)^2 + x^2} \right)^{\frac{n-2}{2}} g(B(x, y)) \tag{4}$$

where B is the Möbius transformation that maps the upper half space $\mathbb{H}^n$ to the unit ball $\mathbb{B}^n$, i.e.,

$$\mathbf{\Omega} = B(x, y) = \frac{(2x, 1 - x^2 - y^2)}{(1+y)^2 + x^2} \ . \tag{5}$$

Inserting (4) into (1) a basically straightforward computation involving some integration by parts yields (3). Clearly, this functional is invariant under rotation. Note that these two representations, the one on the half space and the one on the unit ball show the invariance of the functional under all *Möbius transformations* that preserve the upper half space. This indicates that the term containing the expression $(1 - |\mathbf{\Omega}|^2)^{-2}$ has some intrinsic geometric meaning. A natural way to write the problem (1) is via stereographic projection from the unit ball to the hyperboloid $\mathbb{P}^n$. Once more, set

$$g(\mathbf{\Omega}) = \left(\frac{2}{1 - |\mathbf{\Omega}|^2} \right)^{\frac{n-2}{2}} k(P(\mathbf{u})) \tag{6}$$

where

$$P(\mathbf{u}) = \frac{(2\mathbf{\Omega}, 1 + |\mathbf{\Omega}|^2)}{1 - |\mathbf{\Omega}|^2} \ . \tag{7}$$

It is easy to check that P maps the unit ball to the upper branch of the hyperboloid $u^2 - v^2 = 1$, where $\mathbf{u} = (u, v)$, $u \in \mathbb{R}^n$ and $v \in \mathbb{R}$. Inserting (6) into (3) yields the equivalent inequality

$$\int_{\mathbb{P}^n} |\nabla k(\mathbf{u})|^2 d\mathrm{Vol} - \frac{(n-1)^2}{4} \int_{\mathbb{P}^n} |k(\mathbf{u})|^2 d\mathrm{Vol} \geq C_n \left(\int_{\mathbb{P}^n} |k(\mathbf{u})|^{\frac{2n}{n-2}} d\mathrm{Vol} \right)^{\frac{n-2}{n}} \ . \tag{8}$$

The metric used here on $\mathbb{P}^n$ is the one induced by the Euclidean space $\mathbb{R}^{n+1}$.

As mentioned before the half space problem has been investigated in [10], but in its formulation on the hyperbolic space it has also been investigated before (see [6] for references) although under a different point of view. There one asks whether there exists a constant B_n such that the inequality

$$\int_{\mathbb{P}^n} |\nabla k(\mathbf{u})|^2 d\mathrm{Vol} \geq S_n \left(\int_{\mathbb{P}^n} |k(\mathbf{u})|^{\frac{2n}{n-2}} d\mathrm{Vol} \right)^{\frac{n-2}{n}} + B_n \int_{\mathbb{P}^n} |k(\mathbf{u})|^2 d\mathrm{Vol} \tag{9}$$

holds. Here S_n is the Sobolev constant,

$$\frac{n(n-2)}{4} |\mathbb{S}^n|^{\frac{2}{n}} \tag{10}$$

where $|\mathbb{S}^n|$ is the volume of the n-dimensional unit sphere in $\mathbb{R}^{n+1}$. For $n > 3$ the sharp constant $B_n = \frac{n(n-2)}{4}$ (see [6]). Note that $\frac{n(n-2)}{4} < \frac{(n-1)^2}{4}$. In this language, the problem investigated in [10] is different, i.e., replace B_n by the optimal constant and then find the sharp constant C_n that will replace S_n. Certainly $C_n \leq S_n$, in fact $C_n < S_n$ for $n > 3$. Note that, in this case the exact value of C_n is not known.

In both formulations the interesting case $n = 3$ is conspicuously absent and it is this case we would like to address in this letter. We have

1.1 THEOREM. *The inequality*

$$\int_{\mathbb{H}^3} |\nabla f(\mathbf{x})|^2 d\mathbf{x} \geq \int_{\mathbb{H}^3} \frac{1}{4y^2} |f(\mathbf{x})|^2 d\mathbf{x} + S_3 \left(\int_{\mathbb{H}^3} |f(\mathbf{x})|^6 d\mathbf{x} \right)^{\frac{1}{3}} \tag{11}$$

holds where S_3 is the sharp Sobolev constant in three dimensions, i.e.,

$$S_3 = 3(\pi/2)^{4/3} \ . \tag{12}$$

The inequality is always strict for nonzero f's. Using the formulation on hyperbolic space we have the inequality

$$\int_{\mathbb{P}^3} |\nabla k(\mathbf{u})|^2 d\mathrm{Vol} \geq S_3 \left(\int_{\mathbb{P}^3} |k(\mathbf{u})|^6 d\mathrm{Vol} \right)^{\frac{1}{3}} + \int_{\mathbb{P}^3} |k(\mathbf{u})|^2 d\mathrm{Vol} \ . \tag{13}$$

In contrast to the case $n = 3$, for $n \geq 4$ the sharp constant is always attained for some nonzero function (see [10]).

The problem (1) has been generalized to the case where the underlying domain D is a convex set. In this case one replaces $\frac{1}{4y^2}$ by $\frac{1}{4d(x)^2}$ where $d(x)$ is the distance of the point $x \in D$ to the boundary of D. It is conjectured in [10] that the sharp constant for convex domains is given by the half space problem. This is true for the case where the domain is a ball. We have

1.2 THEOREM. *The inequality*

$$\int_{\mathbb{B}^n} |\nabla g(\mathbf{\Omega})|^2 d\mathbf{\Omega} - \int_{\mathbb{B}^n} \frac{1}{4(1-|\mathbf{\Omega}|)^2} |g(\mathbf{\Omega})|^2 d\mathbf{\Omega} \geq C_n \left(\int_{\mathbb{B}^n} |g(\mathbf{\Omega})|^{\frac{2n}{n-2}} d\mathbf{\Omega} \right)^{\frac{n-2}{n}} \tag{14}$$

holds for all smooth functions compactly supported in the unit ball. For nonzero g's the inequality is always strict.

The inequality follows directly from (3) by noting that for $|\mathbf{\Omega}| < 1$,

$$\frac{1}{(1-|\mathbf{\Omega}|^2)^2} > \frac{1}{4(1-|\mathbf{\Omega}|)^2} \ . \tag{15}$$

That the inequality is sharp and always strict for non-zero functions can be seen by scaling down a compactly supported 'almost' optimizer of the half space problem and use this as a trial function for the ball problem. Note that this device also works for general convex domains. The hard part is to establish the analog of (14) for general convex domains.

An amusing consequence of the formulation (3) is that by inversion with respect to the unit sphere one obtains a sharp inequality on the complement of the unit ball, i.e., we have

1.3 THEOREM. *The inequality*

$$\int_{(\mathbb{B}^n)^c} |\nabla g(\mathbf{\Omega})|^2 d\mathbf{\Omega} - \int_{(\mathbb{B}^n)^c} \frac{1}{(1-|\mathbf{\Omega}|^2)^2} |g(\mathbf{\Omega})|^2 d\mathbf{\Omega} \geq C_n \left(\int_{(\mathbb{B}^n)^c} |g(\mathbf{\Omega})|^{\frac{2n}{n-2}} d\mathbf{\Omega} \right)^{\frac{n-2}{n}} \tag{16}$$

holds for all functions that are smooth and have compact support on $(\mathbb{B}^n)^c$ the complement of the ball $\mathbb{B}^n$ in $\mathbb{R}^n$. Moreover, for $n > 3$ equality can be attained in the sense of [10].

The appropriate formulation of this inequality for general domains, not necessarily convex, is an open problem. Theorem 1.3 suggests that the 'correct' inequality is formulated in terms of either the *harmonic radius* or the *hyperbolic radius* of a domain D. For a definition of these concepts we refer the reader to [1]. Both of these objects are conformally covariant, i.e., under conformal transformations they scale with the n-th root of the Jacobian. In the case of a ball, the two concepts coincide and are equal to $(1-|\mathbf{\Omega}|^2)$. Since the ball and the half space are conformally the same, these two concepts coincide also on the half space and are given by $2y$. Thus, it is natural to ask for which domain D does the inequality

$$\int_D \left[|\nabla f|^2 - \frac{1}{R(x)^2} |f(x)|^2 \right] d^n x \geq C_n \left(\int_D |f(x)|^{\frac{2n}{n-2}} d^n x \right)^{\frac{n-2}{n}} \tag{17}$$

hold. Here $R(x)$ is either the harmonic radius or the hyperbolic radius. In this formulation, due to its conformal invariance, one might be able to show that the Hardy-Sobolev-Maz'ya inequality for general convex domains holds with the same constant as the one on the half space.

The plan of the paper is the following. In Section 2 we derive the Green function for fractional powers of the operator $-\Delta - \frac{1}{4y^2}$. This yields Hardy-Littlewood-Sobolev type kernels. In Section 3 we prove L^p estimates for these kernels and recover Theorem 1.1.

2 The Green function

It is convenient to start with the following heat type equation on the upper half space $\mathbb{H}^n$

$$u_t = \Delta u + \frac{1}{4y^2} u \ , u(x, y; 0) = f(x, y) \ . \tag{18}$$

Substituting $u = \sqrt{y} g$ one obtains the equation

$$g_t = \Delta_x g + g_{yy} + \frac{1}{y} g_y \ , g(x, y; 0) = \frac{f(x, y)}{\sqrt{y}} \ , \tag{19}$$

and one see that the right side of the equation is an $n+1$ dimensional Laplacian. Note that $g_{yy} + \frac{1}{y} g_y$ is the two dimensional Laplacian of a radial function. A similar idea has been used in [2] in a different context. With this in mind one arrives at once at the following formula for the solution of the heat equation

$$u(x, y; t) = \int_{\mathbb{H}^n} G(x - x', y, y'; t) f(x', y') dx' dy' \tag{20}$$

where

$$G(x - x', y, y'; t) = \left(\frac{1}{4\pi t}\right)^{\frac{n+1}{2}} \sqrt{yy'} e^{-\frac{(x-x')^2 + y^2 + y'^2}{4t}} \int_0^{2\pi} e^{\frac{yy'}{2t} \cos\phi} d\phi \ . \tag{21}$$

It is not hard to see that this heat kernel is a contraction semigroup on $L^2(\mathbb{H}^n)$ with Lebesgue measure. Thus, the generator Q is a selfadjoint operator and it is an extension of $-\Delta - \frac{1}{4y^2}$ originally defined on smooth functions with compact support in $\mathbb{H}^n$. Note that the L^2-norm of the gradient of functions in the domain of Q is in general not finite. We shall continue to use the symbol $-\Delta - \frac{1}{4y^2}$ to denote Q.

It is straight forward to see (see e.g., Theorem 7.10 in [8]) that

$$\lim_{t \to 0} \frac{1}{t} \left[\|f\|^2_{L^2(\mathbb{H}^n)} - (f, G_t f)_{L^2(\mathbb{H}^n)} \right] = 2\pi \int_{\mathbb{H}^n} \left(|\nabla_x g|^2 + |g_y|^2 \right) y dy dx \tag{22}$$

where $G_t f$ is the solution of the intial value problem (18) and $g = \frac{f}{\sqrt{y}}$. Note that the right hand side is manifestly positive and coincides with the interpretation of $-\Delta - \frac{1}{4y^2}$ given in [10].

Via the heat kernel it is straightforward to find the kernel of the fractional powers

$$(-\Delta - \frac{1}{4y^2})^{-\frac{\alpha}{2}}(\mathbf{x}; \mathbf{x}') = \frac{1}{\Gamma(\frac{\alpha}{2})} \int_0^\infty t^{\frac{\alpha}{2}} G(x - x', y, y'; t) \frac{dt}{t} \ , \tag{23}$$

for $\alpha > 0$, and a calculation leads to the expression

$$\begin{aligned} & (-\Delta - \frac{1}{4y^2})^{-\frac{\alpha}{2}}(\mathbf{x}; \mathbf{x}') && (24) \\ = \ & 2^{-\alpha} \pi^{-\frac{n+1}{2}} \frac{\Gamma(\frac{n+1-\alpha}{2})}{\Gamma(\frac{\alpha}{2})} \sqrt{yy'} \int_0^{2\pi} \left[(x - x')^2 + y^2 + y'^2 - 2yy' \cos\phi \right]^{-\frac{n+1-\alpha}{2}} d\phi && (25) \\ =: \ & \Phi_{n,\alpha}(\mathbf{x}; \mathbf{x}') \ . && (26) \end{aligned}$$

Similarly, well known expressions hold for $(-\Delta)^{-\frac{\alpha}{2}}$ on $\mathbb{R}^n$ which, for reasons that become clear later, we write in terms of the variables (x, y) as

$$\begin{aligned} & (-\Delta)^{-\frac{\alpha}{2}}(\mathbf{x};\mathbf{x}') && (27)\\ = \;& 2^{-\alpha}\pi^{-\frac{n}{2}}\frac{\Gamma(\frac{n-\alpha}{2})}{\Gamma(\frac{\alpha}{2})}\left[(x-x')^2+(y-y')^2\right]^{-\frac{n-\alpha}{2}} && (28)\\ =: \;& \Psi_{n,\alpha}(\mathbf{x};\mathbf{x}')\ . && (29)\end{aligned}$$

First we state some simple pointwise properties about the kernel $\Phi_{n,\alpha}$.

2.1 LEMMA. *If $n \le \alpha \le n+1$, we have that*

$$\sup_a \Phi_{n,\alpha}(x, y+a; x', y'+a) = \lim_{a\to\infty} \Phi_{n,\alpha}(x, y+a; x', y'+a) \equiv \infty\ . \tag{30}$$

If $n-1 \le \alpha < n$ we have that

$$\sup_a \Phi_{n,\alpha}(x, y+a; x', y'+a) = \lim_{a\to\infty} \Phi_{n,\alpha}(x, y+a; x', y'+a) \equiv \Psi_{n,\alpha}(\mathbf{x};\mathbf{x}')\ . \tag{31}$$

Proof. An elementary calculation shows that

$$\Phi_{n,\alpha}(\mathbf{x};\mathbf{x}') = |\mathbf{x}-\mathbf{x}'|^{-n+\alpha}2^{-\alpha}\pi^{-\frac{n+1}{2}}\frac{\Gamma(\frac{n+1-\alpha}{2})}{\Gamma(\frac{\alpha}{2})}F(A)\ , \tag{32}$$

where

$$A = \frac{\sqrt{yy'}}{|\mathbf{x}-\mathbf{x}'|} \tag{33}$$

and

$$F(A) := \int_{-\pi}^{\pi} \frac{A}{[1+2A^2(1-\cos(\phi))]^{\frac{n+1-\alpha}{2}}}d\phi\ . \tag{34}$$

All the statements are an immediate consequence of Lemma 4.1 with $\beta = \frac{n+1-\alpha}{2}$. □

3 L^p-estimates for fractional powers

As a consequence of Lemma 2.1 and Lieb's sharp constant in the Hardy-Littlewood Sobolev inequality [7] we have the following corollary.

3.1 COROLLARY. *If $n \le \alpha \le n+1$ then the operator*

$$(-\Delta - \frac{1}{4y^2})^{-\frac{\alpha}{2}} \tag{35}$$

is not bounded on $L^p(\mathbb{H}^n)$ for any $1 \le p \le \infty$. If $n-1 \le \alpha < n$ then this operator is a bounded operator from $L^p(\mathbb{H}^n)$ to $L^q(\mathbb{H}^n)$ for all $1 < p, q < \infty$ that satisfy

$$\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{n}\ . \tag{36}$$

Moreover, for such values of α we have

$$(f,(-\Delta-\frac{1}{4y^2})^{-\frac{\alpha}{2}}f)\le 2^{-\alpha}\pi^{-\frac{n}{2}}\frac{\Gamma(\frac{n-\alpha}{2})}{\Gamma(\frac{\alpha}{2})}C(n,\alpha)\|f\|_p^2 \tag{37}$$

where $p=\frac{2n}{n+\alpha}$ and

$$C(n,\alpha)=\pi^{\frac{n-\alpha}{2}}\frac{\Gamma(\frac{\alpha}{2})}{\Gamma(\frac{n+\alpha}{2})}\left[\frac{\Gamma(\frac{n}{2})}{\Gamma(n)}\right]^{-\frac{\alpha}{n}} \tag{38}$$

is the sharp constant. This constant is not attained in (37) for nonzero functions.

Proof of Theorem 1.1. We write

$$|(f,g)|=|(Q^{\alpha/4}f,Q^{-\alpha/4}g)|\le(f,Q^{\alpha/2}f)^{1/2}(g,Q^{-\alpha/2}g)^{1/2} \tag{39}$$

which by Corollary 3.1 yields the bound

$$|(f,g)|^2\le 2^{-\alpha}\pi^{-\frac{n}{2}}\frac{\Gamma(\frac{n-\alpha}{2})}{\Gamma(\frac{\alpha}{2})}C(n,\alpha)(f,Q^{\alpha/2}f)\|g\|_p^2 \tag{40}$$

for $n-1\le\alpha<n$ and $p=\frac{2n}{n+\alpha}$. Thus,

$$\|f\|_{p'}^2<2^{-\alpha}\pi^{-\frac{n}{2}}\frac{\Gamma(\frac{n-\alpha}{2})}{\Gamma(\frac{\alpha}{2})}C(n,\alpha)(f,Q^{\alpha/2}f)\ , \tag{41}$$

and there is never equality in the above inequality for nonzero functions. Theorem 1.1 follows by choosing $n=3$ and $\alpha=2$. $\square$

The reader may wonder what happens when $0<\alpha<n-1$. While we do not succeed in calculating the sharp constant, it is possible to show that the sharp constant in inequality (37) is attained. While this constant is strictly bigger than the corresponding constant in the Hardy-Littlewood-Sobolev inequality, we do not know its exact value.

The procedure for proving this relies on the conformal invariance of the kernel which allows to transform the problem into one on the unit ball. Then the device of competing symmetries developed in [4] allows to restrict the maximization problem to radial functions on the ball. The correction term to Fatou's lemma ([3], see also [8]) then allows to show the existence of a maximizer. Thus, we recover some of the results in [10] with a different proof. Moreover, it is also possible to show that every maximizer is the conformal image of a radial function. The details will appear elsewhere.

4 Appendix

In this appendix we collect some facts about the function

$$F(A):=\int_{-\pi}^{\pi}\frac{A}{(1+2A^2(1-\cos(\phi))^{\beta}}d\phi\ , \tag{42}$$

where $\beta=\frac{n+1-\alpha}{2}$.

4.1 LEMMA. *Depending on the value of β, the function $F(A)$ has the following asymptotics as $A \to \infty$.*

a) If $0 \leq \beta \leq \frac{1}{2}$ then $\lim_{A\to\infty} F(A) = \infty$.

b) If $\frac{1}{2} < \beta \leq 1$, then $F(A)$ is a monotone increasing function and

$$\lim_{A\to\infty} F(A) = \sqrt{\pi}\frac{\Gamma(\beta - \frac{1}{2})}{\Gamma(\beta)} \ . \tag{43}$$

Proof. Since

$$F(A) = \int_{-\pi A}^{\pi A} \frac{1}{1 + 2A^2(1 - \cos(\frac{\phi}{A}))^\beta} d\phi \tag{44}$$

the limit as $A \to \infty$ is

$$\int_{-\infty}^{\infty} \frac{1}{(1 + \phi^2)^\beta} d\phi = \sqrt{\pi}\frac{\Gamma(\beta - \frac{1}{2})}{\Gamma(\beta)} \tag{45}$$

for $\beta > \frac{1}{2}$ and it is $+\infty$ for $\beta \leq \frac{1}{2}$. This proves a). To see that b) holds for $\beta = 1$ one easily performs the ϕ integration and obtains

$$F(A) = \frac{2\pi A}{\sqrt{1 + 4A^2}} \ , \tag{46}$$

which is obviously increasing with A. For $\frac{1}{2} < \beta < 1$ we use the formula

$$\begin{aligned} &\left[1 + 2A^2(1 - \cos\phi)\right]^{-\beta} &(47)\\ = \ &\frac{\sin(\pi\beta)}{\pi} \int_0^\infty \left[1 + t + 2A^2(1 - \cos\phi)\right]^{-1} t^{1-\beta} \frac{dt}{t} \ . &(48) \end{aligned}$$

Integrating with respect to ϕ yields

$$F(A) = 2\sin(\pi\beta) \int_0^\infty \frac{A}{\sqrt{(1 + t)^2 + 4(1 + t)A^2}} t^{1-\beta} \frac{dt}{t} \ . \tag{49}$$

Again, this function increases with A. $\square$

References

[1] Bandle, C. and Flucher, M., Harmonic radius and concentration of energy; hyperbolic radius and Liouville's equations $\Delta U = e^U$ and $\Delta U = U^{\frac{n+2}{n-2}}$, SIAM Review, **38** (1996), 191–238.

[2] Beckner, W., On the Grushin operator and hyperbolic symmetry, Proc. Amer. Math. Soc. **129** (2000), 1233–1246.

[3] Brézis, H. and Lieb, E.H., A relation between pointwise convergence of functions and convergence of functionals, Proc. Amer. Math. Soc. **88** (1983), 486–490.

[4] Carlen, E.A. and Loss, M., On the minimization of symmetric functionals, Rev. Math. Phys. **6**, (1994), 1011–1032.

[5] Filippas, S., Maz'ya, V. G. and Tertikas, A., Sharp Hardy–Sobolev inequalities, C. R. Math. Acad. Sci. Paris **339** (2004), no. 7, 483–486.

[6] Hebey, E., *Nonlinear Analysis on Manifolds: Sobolev Spaces and Inequalities*, Courant Lecture Notes in Mathematics Vol. 5, AMS, Providence, RI, 1999.

[7] Lieb, E.H., Sharp constants in the Hardy–Littlewood–Sobolev and related inequalities, Ann. Math. **118** (1983), 349–374.

[8] Lieb, E.H. and Loss, M., *Analysis*, Graduate Studies in Mathematics Vol. 14, AMS, Providence, 2001.

[9] Maz'ya, V.G., *Sobolev Spaces*, Springer Verlag, Berlin New York, 1985.

[10] Tertikas, A. and Tintarev, K., On existence of minimizers for the Hardy–Sobolev–Maz'ya inequality, published online in Ann. Mat. Pura Appl., 7 September 2006.